Aggregation of Votes with Multiple Positions on Each Issue *

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We consider the problem of aggregating votes cast by a society on a fixed set of issues, where each member of the society may vote for one of several positions on each issue, but the combination of votes on the various issues is restricted to a set of feasible voting patterns. We require the aggregation to be supportive, i.e., for every issue, the corresponding component of every aggregator, when applied to a tuple of votes, must take as value one of the votes in that tuple. We prove that, in such a set-up, non-dictatorial aggregation of votes in a society of an arbitrary size is possible if and only if a non-dictatorial binary aggregator exists or a non-dictatorial ternary aggregator exists such that, for each issue, the corresponding component of the aggregator, when restricted to two-element sets of votes, is a majority operation or a minority operation. We then introduce a notion of a uniform non-dictatorial aggregator, which is an aggregator such that on every issue, and when restricted to arbitrary two-element subsets of the votes for that issue, differs from all projection functions. We first give a characterization of sets of feasible voting patterns that admit a uniform non-dictatorial aggregator. After this and by making use of Bulatov's dichotomy theorem for conservative constraint satisfaction problems, we connect social choice theory with the computational complexity of constraint satisfaction by proving that if a set of feasible voting patterns has a uniform non-dictatorial aggregator of some arity, then the multi-sorted conservative constraint satisfaction problem on that set (with each issue representing a different sort) is solvable in polynomial time; otherwise, it is NP-complete.

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1 INTRODUCTION

Kenneth Arrow initiated the theory of aggregation by establishing his celebrated General Possibility Theorem (also known as Arrow's Impossibility Theorem) [1], which asserts that it is impossible, even under mild conditions, to aggregate in a non-dictatorial way the preferences of a society. Wilson [15] introduced aggregation on general attributes, rather than just preferences, and proved Arrow's result in this context. Later on, Dokow and Holzman [7] adopted a framework similar to Wilson's in which the voters have a binary position on a number of issues, and an individual voter's

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feasible position patterns are restricted to lie in a domain X. Dokow and Holzman discovered a necessary and sufficient condition for X to have a non-dictatorial aggregator that involves a property called *total blockedness*, which was originally introduced in [9]. Roughly speaking, a domain X is totally blocked if "any position on any issue can be deduced from any position on any issue" (the precise definition is given in Section 3). In other words, total blockedness is a property that refers to the propagation of individuals' positions from one issue to another.

After this, Dokow and Holzman [8] extended their earlier work by allowing the positions to be non-Boolean (non-binary). By generalizing the notion of a domain being totally blocked to the non-Boolean framework, they gave a sufficient (but not necessary) condition for non-dictatorial aggregation, namely, they showed that if a domain is not totally blocked, then it is a possibility domain. Recently, Szegedy and Xu [13] discovered necessary and sufficient conditions for non-dictatorial aggregation. Quite remarkably, their approach relates aggregation theory with universal algebra, specifically with the structure of the space of *polymorphisms*, that is, functions under which a relation is closed. It should be noted that properties of polymorphisms have been successfully used towards the delineation of the boundary between tractability and intractability for the Constraint Satisfaction Problem (for an overview, see, e.g., [6]).

Szegedy and Xu [13] distinguished the *supportive* (also known as *conservative*) case, where the social position must be equal to the position of at least one individual, from the *idempotent* (also known as *Paretian*) case, where the social position need not agree with any individual position, unless the votes are unanimous. In the idempotent case, they gave a necessary and sufficient condition for possibility of non-dictatorial aggregation that involves no propagation criterion (such as the domain being totally blocked), but only refers to the possibility of non-dictatorial aggregation for societies of a fixed cardinality (as large as the space of positions). In the supportive case, however, their necessary and sufficient conditions still involve the notion of the domain being totally blocked.

Here, we follow Szegedy and Xu's idea of deploying the algebraic "toolkit" [13] and we prove that, in the supportive case, non-dictatorial aggregation is possible for all societies of some cardinality if and only if a non-dictatorial binary aggregator exists or a non-dictatorial ternary aggregator exists such that on every issue j, the corresponding component f_j is a majority operation, i.e., for all x and y, it satisfies the equations

$$f_j(x,x,y) = f_j(x,y,x) = f_j(y,x,x) = x$$

or f_j is a minority operation, i.e., for all x and y, it satisfies the equations

$$f_j(x,x,y) = f_j(x,y,x) = f_j(y,x,x) = y.$$

(For additional information about the notions of majority and minority operations, see Szendrei [14, p. 24].)

We also show that a domain is totally blocked if and only if it admits no non-dictatorial binary aggregator; thus, the notion of a domain being totally blocked is, in a precise sense, a weak form of an impossibility domain.

After this. we introduce the notion of *uniform* non-dictatorial aggregator, which is an aggregator that on every issue, and when restricted to an arbitrary two-element subset of the votes for that issue, differs from all projection functions. We first give a characterization of sets of feasible voting patterns that admit uniform non-dictatorial aggregators. Then, making use of Bulatov's dichotomy theorem for conservative constraint satisfaction problems (see [2–4]), we connect social choice theory with the computational complexity of constraint satisfaction by proving that if a set of feasible voting patterns X has a uniform non-dictatorial aggregator of some arity, then the multi-sorted conservative constraint satisfaction problem on X, in the sense introduced by Bulatov and Jeavons [5], with each issue representing a sort, is tractable; otherwise it is NP-complete.

2 BASIC CONCEPTS AND EARLIER WORK

2.1 Basic Concepts

In all that follows, we have a fixed set $I = \{1, \dots, m\}$ of issues. Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a family of finite sets, each of cardinality at least 2, representing the possible positions (voting options) on the issues $1, \dots, m$, respectively. If every A_j has cardinality exactly 2 (i.e., if for every issue only a "yes" or "no" vote is allowed), we say that we are in the *binary* or the *Boolean framework*; otherwise, we say that we are in the *non-binary* or the *non-Boolean* framework.

Product?

Let X be a non-empty subset of $\prod_{j=1}^{m} A_j$ that represents the feasible voting patterns. We write X_j, $j = 1 \dots, m$, to denote the j-th projection of X. From now on, we assume that each X_j has cardinality at least 2 (this is a *non-degeneracy* condition). Throughout the rest of the paper, unless otherwise declared, X will denote a set of feasible voting patterns on m issues, as we just described.

Let $n \geq 2$ be an integer representing the number of voters. The elements of X^n can be viewed as $n \times m$ matrices, whose rows correspond to voters and whose columns correspond to issues. We write x_j^i to denote the entry of the matrix in row i and column j; clearly, it stands for the vote of voter i on issue j. The row vectors of such matrices will be denoted as $x^1, \dots, x^n$, and the column vectors as $x_1, \dots, x_m$.

Let now $\bar{f} = (f_1, \dots, f_m)$ be an m-tuple of n-ary functions $f_j : A_j^n \mapsto A_j$.

An m-tuple of functions $\bar{f} = (f_1, \dots, f_m)$ as above is called *supportive (conservative)* if for all $j = 1 \dots m$, we have that:

$$\text{if } x_j = (x_j^1, \dots, x_j^n) \in A_j^n, \text{ then } f_j(x_j) = f_j(x_j^1, \dots, x_j^n) \in \{x_j^1, \dots, x_j^n\}. \quad ?$$

An m-tuple $\bar{f} = (f_1, \dots, f_m)$ of (n-ary) functions as above is called an (n-ary) *aggregator for* X if it is supportive and, for all $j = 1, \dots, m$ and for all $x_j \in A_j^n, j = 1, \dots, m$, we have that:

$$\text{if } (x^1, \dots, x^n) \in X^n, \text{ then } (f_1(x_1), \dots, f_m(x_m)) \in X.$$

Note that $(x^1, \dots, x^n)$ is an $n \times m$ matrix with rows $x^1, \dots, x^n$ and columns $x_1, \dots, x_m$, whereas $(f_1(x_1), \dots, f_m(x_m))$ is a row vector required to be in X. The fact that aggregators are defined as m-tuples of functions $A_j^n \mapsto A_j$, rather than a single function $X^n \mapsto X$, reflects the fact that the social vote is assumed to be extracted issue-by-issue, i.e., the aggregate vote on each issue does not depend on voting data on other issues.

An aggregator $\bar{f} = (f_1, \dots, f_m)$ is called *dictatorial on* X if there is a number $d \in \{1, \dots, n\}$ such that $(f_1, \dots f_m) \restriction X = (\mathrm{pr}_d^n, \dots, \mathrm{pr}_d^n) \restriction X$, i.e., $(f_1, \dots f_m)$ restricted to X is equal to $(\mathrm{pr}_d^n, \dots, \mathrm{pr}_d^n)$ restricted to X, where pr_d^n is the n-ary projection on the d-th coordinate; otherwise, $\bar{f}$ is called *non-dictatorial on* X. We say that X *has a non-dictatorial aggregator* if, for some $n \geq 2$, there is a non-dictatorial n-ary aggregator on X.

A set X of feasible voting patterns is called a *possibility domain* if it has a non-dictatorial aggregator. Otherwise, it is called an *impossibility domain*. A possibility domain is, by definition, one where aggregation is possible for societies of some cardinality, namely, the arity of the non-dictatorial aggregator.

Aggregators do what their name indicates, that is, they aggregate positions on m issues, $j = 1, \dots, m$, from data representing the voting patterns of n individuals on all issues. The fact that aggregators are assumed to be supportive (conservative) reflects the restriction of our model that the social vote for every issue should be equal to the vote cast on this issue by at least one individual.

Finally, the requirement of non-dictatorialness for aggregators reflects the fact that the aggregate vote should not be extracted by adopting the vote of a single individual designated as a "dictator".

Example 2.1. Suppose that X is a cartesian product $X = Y \times Z$, where $Y \subseteq \prod_{j=1}^{l} A_j$ and $Z \subseteq \prod_{j=l+1}^{m} A_j$, with $1 \leq l < m$. It is easy to see that X is a possibility domain.

Indeed, for every $n \geq 2$, the set X has non-dictatorial n-ary aggregators of the form $(f_1, \ldots, f_l, f_{l+1}, \ldots, f_m)$, where for some d and d' with $d \neq d'$, we have $f_j = \mathrm{pr}_d^n$, for $j = 1, \ldots, l$, and also $f_j = \mathrm{pr}_{d'}^n$, for $j = l+1, \ldots, m$. Thus, every cartesian product of two sets of feasible patterns is a possibility domain. □

Now, following Á. Szendrei [14, p. 24], we define the notions of a majority operation and of a minority operation.

Definition 2.2. A ternary operation $f : A^3 \mapsto A$ on an arbitrary set A is a *majority* operation if for all x and y in A,

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = x,$$

and it is a *minority* operation if for all x and y in A,

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = y.$$

We also define what it means for a set to admit a majority operation and a minority operation. (Since the arity of an aggregator is the arity of its component functions, a ternary aggregator is an aggregator with components of arity three.)

Definition 2.3. Let X be a set of feasible voting patterns.

- X admits a *majority aggregator* if it admits a ternary aggregator $\bar{f} = (f_1, \ldots, f_m)$ such that f_j is a majority operation on X_j, for all $j = 1, \ldots, m$.
- X admits a *minority aggregator* if it admits a ternary aggregator $\bar{f} = (f_1, \ldots, f_m)$ such that f_j is a minority operation on X_j, for all $j = 1, \ldots, m$.

Clearly, X admits a majority aggregator if and only if there is a ternary aggregator $\bar{f} = (f_1, \ldots, f_m)$ for X such that, for all $j = 1, \ldots, m$ and for all two-element subsets $B_j \subseteq X_j$, we have that $f_j \restriction B_j = \mathrm{maj}$, where

$$\mathrm{maj}(x, y, z) = \begin{cases} x & \text{if } x = y \text{ or } x = z, \\ y & \text{if } y = z. \end{cases}$$

Also, X admits a minority aggregator if and only if there is a ternary aggregator $\bar{f} = (f_1, \ldots, f_m)$ for X such that, for all $j = 1, \ldots, m$ and for all two-element subsets $B_j \subseteq X_j$, we have that $f_j \restriction B_j = \oplus$, where

$$\oplus(x, y, z) = \begin{cases} z & \text{if } x = y, \\ x & \text{if } y = z, \\ y & \text{if } x = z. \end{cases}$$

It is known that in the Boolean framework (in which for all issues only "yes" or "no" votes are allowed), a set X admits a majority aggregator if and only if X is a bijunctive logical relation, i.e., a subset of $\{0, 1\}^m$ that is the set of satisfying assignments of a 2CNF-formula. Moreover, X admits a minority aggregator if and only if X is an affine logical relation, i.e., a subset of $\{0, 1\}^m$ that is the set of solutions of linear equations over the two-element field (see Schaefer [12]).

Example 2.4. The set $X = \{(a, a, a), (b, b, b), (c, c, c), (a, b, b), (b, a, a), (a, a, c), (c, c, a)\}$ admits a majority aggregator.

To see this, let $\bar{f} = (f, f, f)$, where $f : \{a, b, c\} \to \{a, b, c\}$ is as follows:

$$f(u, v, w) = \begin{cases} a & \text{if } u,\ v, \text{ and } w \text{ are pairwise different;} \\ \mathrm{maj}(u, v, w) & \text{otherwise.} \end{cases}$$

Clearly, if B is a two-element subset of $\{a, b, c\}$, then $f \upharpoonright B = \mathrm{maj}$. So, to show that X admits a majority aggregator, it remains to show that $\bar{f} = (f, f, f)$ is an aggregator for X. In turn, this amounts to showing that $\bar{f}$ is supportive and that X is closed under f. It is easy to check that $\bar{f}$ is supportive. To show that X is closed under f, let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), z = (z_1, z_2, z_3)$ be three elements of X. We have to show that $(f(x_1, y_1, z_1), f(x_2, y_2, z_2), f(x_3, y_3, z_3))$ is also in X. The only case that needs to be considered is when x, y, and z are pairwise distinct. Several subcases need to be considered. For instance, if $x = (a, b, b)$, $y = (a, a, c)$, $z = (c, c, a)$, then $\bar{f}(x, y, z) = (f(a, a, c), f(b, a, c), f(b, c, a)) = (a, a, a) \in X$; the remaining combinations are left to the reader. □

Example 2.5. The set $X = \{(a, b, c), (b, a, a), (c, a, a)\}$ admits a minority aggregator.

To see this, let $\bar{f} = (f, f, f)$, where $f : \{a, b, c\} \to \{a, b, c\}$ is as follows:

$$f(u, v, w) = \begin{cases} a & \text{if } u,\ v, \text{ and } w \text{ are pairwise different;} \\ \oplus(u, v, w) & \text{otherwise.} \end{cases}$$

Clearly, if B is a two-element subset of $\{a, b, c\}$, then $f \upharpoonright B = \oplus$. So, to show that X admits a minority aggregator, it remains to show that $\bar{f} = (f, f, f)$ is an aggregator for X. In turn, this amounts to showing that $\bar{f}$ is supportive and that X is closed under f. It is easy to check that $\bar{f}$ is supportive. To show that X is closed under f, let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), z = (z_1, z_2, z_3)$ be three elements of X. We have to show that $(f(x_1, y_1, z_1), f(x_2, y_2, z_2), f(x_3, y_3, z_3))$ is also in X. The only case that needs to be considered is when x, y, and z are distinct, say, $x = (a, b, c)$, $y = (b, a, a)$, $z = (c, a, a)$. In this case, we have that $(f(a, b, c), f(b, a, a), f(c, a, a)) = (a, b, c) \in X$; Since f is not affected by permutations of the input, the proof is complete. □

So far, we have given examples of possibility domains only. Next, we give an example of an impossibility domain in the Boolean framework.

Example 2.6. Let $W = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ be the 1-in-3 relation, i.e., the set of all Boolean tuples of length 3 in which exactly one 1 occurs.

We claim that W is an impossibility domain. It is not hard to show that W is not affine and that it does not admit a non-dictatorial binary aggregator. Theorem 3.7 in the next section implies that W is an impossibility domain. □

Every logical relation $X \subseteq \{0, 1\}^m$ gives rise to a generalized satisfiability problem in the context studied by Scheafer [12]. We point out that the property of X being a possibility domain in the Boolean framework is not related to the tractability of the associated generalized satisfiability problem. Concretely, the set W in Example 2.6 is an impossibility domain and its associated generalized satisfiability problem is the NP-complete problem Positive 1-in-3-Sat. As discussed earlier, the cartesian product $W \times W$ is a possibility domain. Using the results in [12], however, it can be verified that the generalized satisfiability problem arising from $W \times W$ is NP-complete. At the same time, the set $\{0, 1\}^m$ is trivially a possibility domain and gives rise to a trivially tractable satisfiability problem. Thus, the property of X being a possibility domain is not related to the tractability of the generalized satisfiability problem arising from X.

Nonetheless, in Section 3 we establish the equivalence between the stronger notion of X being a *uniform possibility domain* and the weaker notion of the tractability of the *multi-sorted* generalized satisfiability problem arising from X, where each issue is taken as a different sort. Actually, we establish this equivalence not only for satisfiability problems but also for constraint satisfaction problems whose variables range over arbitrary finite sets.

2.2 Earlier Work

There has been a significant body of earlier work on possibility domains. Here, we summarize some of the results that relate the notion of a possibility domain to the notion of a set being *totally blocked*, a notion originally introduced in the context of the Boolean framework by Nehring and Puppe [9]. As stated earlier, a set X of possible voting patterns is totally blocked if, intuitively, "any position on any issue can be deduced from any position on any issue"; this intuition is formalized by asserting that a certain directed graph G_X associated with X is strongly connected. The precise definition of this notion is given in Section 3.

In the case of the Boolean framework, Dokow and Holzman [7] obtained the following necessary and sufficient condition for a set to be a possibility domain.

Theorem A (Dokow and Holzman [7, Theorem 2.2]). *Let $X \subseteq \{0, 1\}^m$ be a set of feasible voting patterns. The following statements are equivalent.*

- *X is a possibility domain.*
- *X is affine or X is not totally blocked.*

For the non-Boolean framework, Dokow and Holzman [8] found the following connection between the notions of totally blocked and possibility domain.

Theorem B (Dokow and Holzman [8, Theorem 2]). *Let X be a set of feasible voting patterns. If X is not totally blocked, then X is a possibility domain; in fact, there is a non-dictatorial n-ary aggregator, for every $n \geq 2$.*

Note that, in the case of the Boolean framework, Theorem B was stated and proved as Claim 3.6 in [7].

For the non-Boolean framework, Szegedy and Xu [13] obtained a sufficient and necessary condition for a totally blocked set X to be a possibility domain.

Theorem C (Szegedy and Xu [13, Theorem 8]). *Let X be a set of feasible voting patterns that is totally blocked. The following statements are equivalent.*

- *X is a possibility domain.*
- *X admits a binary non-dictatorial aggregator or a ternary non-dictatorial aggregator.*

Note that, in the case of the Boolean framework, Theorem C follows from the preceding Theorem A (Theorem 2.2 in [7]).

A binary non-dictatorial aggregator can also be viewed as a ternary one, where one of the arguments is ignored. By considering whether or not X is totally blocked, Theorems B and C imply the following corollary, which characterizes possibility domains without involving the notion of total blockedness; to the best of our knowledge, this result has not been explicitly stated previously.

Corollary 2.7. *Let X be a set of feasible voting patterns. The following statements are equivalent.*

(1) X is a possibility domain.
(2) X has a non-dictatorial binary aggregator or a non-dictatorial ternary aggregator.
(3) X has a non-dictatorial ternary aggregator.

3 CHARACTERIZATION OF POSSIBILITY DOMAINS

Our first result is a necessary and sufficient condition for a set of feasible voting patterns to be a possibility domain.

THEOREM 3.1. *Let X be a set of feasible voting patterns. The following statements are equivalent.*

(1) X is a possibility domain.
(2) X has a non-dictatorial binary aggregator or it admits a majority aggregator or it admits a minority aggregator.

Theorem 3.1 is stronger than the preceding Corollary 2.7 because, unlike Corollary 2.7, it gives explicit information about the nature of the components f_j of non-dictatorial ternary aggregators $\bar{f} = (f_1, \ldots, f_m)$, when the components are restricted to a two-element subset $B_j \subseteq X_j$ of the set of positions on issue j, information that is necessary to relate results in aggregation theory with complexity theoretic results (besides the three projections, there are 61 supportive ternary functions on a two element set). Observe also that if $\bar{f} = (f_1, \ldots, f_m)$ is a binary aggregator, then every component f_j is necessarily a projection function or the function $\wedge$ or the function $\vee$, when restricted to a two-element subset $B_j \subseteq X_j$ (identified with the set $\{0, 1\}$). So, for binary aggregators, the information about the nature of their components is given *gratis*.

Only the direction $1 \implies 2$ of Theorem 3.1 requires proof. Towards this goal, we first introduce a new notion, that of monomorphic aggregators, and give three lemmas, which we then use to prove Theorem 3.1.

Let X be a set of feasible voting patterns and let $\bar{f} = (f_1, \ldots, f_m)$ be an n-ary aggregator for X.

Definition 3.2. We say that $\bar{f}$ is *locally monomorphic* if for all indices i and j with $1 \leq i, j \leq m$, for all two-element subsets $B_i \subseteq X_i$ and $B_j \subseteq X_j$, for every bijection $g : B_i \mapsto B_j$, and for all column vectors $x_i = (x_i^1, \ldots, x_i^n) \in B_i^n$, we have that

$$f_j(g(x_i^1), \ldots, g(x_i^n)) = g(f_i(x_i^1, \ldots, x_i^n)).$$

either function or bijection

Intuitively, the above definition says that, no matter how we identify the two elements of B_i and B_j with 0 and 1, the restrictions $f_i \restriction B_i$ and $f_j \restriction B_j$ are equal as functions. Notice that in the definition we are allowed to have $i = j$, which implies that if in a specific B_j we interchange the values 0 and 1 in the arguments of $f_j \restriction B_j$, then the bit that gives the image of $f_j \restriction B_j$ is flipped.

It follows immediately from the definitions that if an aggregator is dictatorial, then it is locally monomorphic. For binary aggregators, the converse is true. Indeed, assume that $\bar{f} = (f_1, \ldots, f_m)$ is a binary locally monomorphic aggregator for X. We claim that $\bar{f} = (f_1, \ldots, f_m)$ is dictatorial on X. To see this, fix a coordinate f_i and consider a pair $(a, b) \in X_i^2$ with $a \neq b$. By conservativeness, either $f_i(a, b) = a$ or $f_i(a, b) = b$. We claim that if $f_i(a, b) = a$, then $(f_1, \ldots, f_m) \restriction X = (\mathrm{pr}_1^2, \ldots, \mathrm{pr}_1^2) \restriction X$, while if $f_i(a, b) = b$, then $(f_1, \ldots, f_m) \restriction X = (\mathrm{pr}_2^2, \ldots, \mathrm{pr}_2^2) \restriction X$. To see this, consider a coordinate f_j and a pair $(a', b') \in X_j^2$ with $a' \neq b'$. Let $g : \{a, b\} \to \{a', b'\}$ be the bijection $g(a) = a'$ and $g(b) = b'$. Since $\bar{f} = (f_1, \ldots, f_m)$ is locally monomorphic, we have that $f_j(a', b') = f_j(g(a), g(b)) = g(f_i(a, b)) = g(a) = a'$, hence $(f_1, \ldots, f_m) \restriction X = (\mathrm{pr}_1^2, \ldots, \mathrm{pr}_1^2) \restriction X$. The case where $f_i(a, b) = b$ is entirely analogous. As we shall see next, a ternary locally monomoprhic aggregator need not be dictatorial. In fact, majority aggregators and minority aggregators are locally monomorphic, but, of course, they are not dictatorial.

Example 3.3. Let X be a set of feasible voting patterns that admits a ternary aggregator $\bar{f} = (f_1, \ldots, f_m)$ that is either a majority or a minority aggregator. Then $\bar{f} = (f_1, \ldots, f_m)$ is locally monomorphic.

Indeed, suppose that $\bar{f} = (f_1, ..., f_m)$ is a minority aggregator, i.e. for every j with $1 \leq j \leq m$ and every two-element set $B_j \subseteq X_j$, we have that $f_j \restriction_{B_j} = \oplus$. Let i, j be such that $1 \leq i, j, \leq m$, let $B_i = \{a, b\} \subseteq X_i$, and let $B_j = \{c, d\} \subseteq X_j$ (we make no assumption for the relation, if any, between a, b, c, d). There are exactly two bijections g and g' from B_i to B_j, namely,

$$g(a) = c \;\&\; g(b) = d$$

$$g'(a) = d \;\&\; g'(b) = c$$

Suppose that (x, y, z) is a triiple with $x, y, z \in B_i$. Since $|B_i| = |B_j| = 2$, it holds that $f_i \restriction_{B_i} = \oplus$ and $f_j \restriction_{B_j} = \oplus$. Without loss of generality, suppose that $x = a, y = z = b$. Then

$$\begin{aligned} f_j(g(x), g(y), g(z)) &= f_j(c, d, d) \\ &= \oplus(c, d, d) = c \\ &= g(a) = g(\oplus(a, b, b)) \\ &= g(f_i(x, y, z)). \end{aligned}$$

An analogous statement holds for g'. Since i, j were arbitrary, we conclude that $\bar{f}$ is locally monomorphic.

The proof for the case when $\bar{f}$ is a majority aggregator is similar. □

We now present the first lemma needed in the proof of Theorem 3.1, which gives a sufficient condition for all aggregators of all arities to be locally monomorphic.

Lemma 3.4. *Let X be a set of feasible voting patterns. If every binary aggregator for X is dictatorial on X, then, for every $n \geq 2$, every n-ary aggregator for X is locally monomorphic.*

Proof. Under the hypothesis that all binary aggregators are dictatorial, the conclusion is obviously true for binary aggregators. By induction, suppose that the conclusion is true for all $(n-1)$-ary aggregators, where $n \geq 3$. Consider an n-ary aggregator $\bar{f} = (f_1, \ldots, f_m)$ and a pair (B_i, B_j) of two-element subsets $B_i \subseteq X_i$ and $B_j \subseteq X_j$. To render the notation less cumbersome, we will take the liberty to denote the two elements of both B_i and B_j as 0 and 1. Assume now, towards a contradiction, that there are a column-vector $(a^1, \ldots, a^n)$ with $a^i \in \{0, 1\}$, $1 \leq i \leq n$, a "copy" of this vector belonging to B_i^n, another copy belonging to B_j^n, such that $f_i(a_1, \ldots, a^n) \neq f_j(a^1, \ldots, a^n)$. Since $n \geq 3$, by the pigeonhole principle applied to two holes and at least three pigeons, there is a pair of coordinates of $(a^1, \ldots, a^n)$ that coincide. Without loss of generality, assume that these two coordinates are the two last ones, i.e., $a^{n-1} = a^n$. We now define an $(n-1)$-ary aggregator $\bar{g} = (g_1, \ldots, g_m)$ as follows: given $n-1$ voting patterns $(x_1^i, \ldots, x_m^i)$, $i = 1, \ldots, n-1$, define n voting patterns by just repeating the last one and then for all $k = 1, \ldots, m$, define

$$g_k(x_k^1, \ldots, x_k^{n-1}) = f_k(x_k^1, \ldots, x_k^{n-1}, x_k^{n-1}).$$

It is straightforward to verify that $\bar{g}$ is an $(n-1)$-ary aggregator on X that is not locally monomorphic, which contradicts the inductive hypothesis. □

Remark 1. The preceding argument generalizes to arbitrary cardinalities in the following way: if every aggregator of arity at most s on X is dictatorial, then every aggregator on X is *s-locally monomorphic*, meaning that for every $k \leq s$ and for all sets $B_j \subseteq X_j$ of cardinality k, the functions $f_j \restriction B_j$ are all equal up to bijections between the B_j's.

Next, we state a technical lemma whose proof was inspired by a proof in Dokow and Holzman [8, Proposition 5].

dictator supportive / aggregator

Lemma 3.5. *Assume that for all integers $n \geq 2$ and for every n-ary aggregator $\bar{f} = (f_1, \ldots, f_m)$, there is an integer $d \leq n$ such that for every integer $j \leq m$ and every two-element subset $B_j \subseteq X_j$, the restriction $f_j \upharpoonright B_j$ is equal to pr^n_d, the n-ary projection on the d-th coordinate. Then for all integers $n \geq 2$ and for every n-ary aggregator $\bar{f} = (f_1, \ldots, f_m)$ and for all $s \geq 2$, there is an integer $d \leq n$ such that for every integer $j \leq m$ and every subset $B_j \subseteq X_j$ of cardinality at most s, the restriction $f_j \upharpoonright B_j$ is equal to pr^n_d.*

$\equiv d$

Proof. The proof will be given by induction on s. The induction basis $s = 2$ is given by hypothesis. Before delving into the inductive step of the proof and for the purpose of making the intuition behind it clearer, let us mention the following fact whose proof is left to the reader. This fact illustrates the idea for obtaining a non-dictatorial aggregator of lower arity from one of higher arity.

Fact. Let A be a set and let $f : A^3 \mapsto A$ be a supportive function such that if among x_1, x_2, x_3 at most two are different, then $f(x_1, x_2, x_3) = x_1$. Assume also that there exist pairwise distinct a_1, a_2, a_3 such that $f(a_1, a_2, a_3) = a_2$; in the terminology of universal algebra, f is a *semi-projection*, but not a projection. Define $g(x_1, x_2) = f(x_1, f(x_1, x_2, a_3), a_3)$. Then, by distinguishing cases as to the value of $f(x_1, x_2, a_3)$, it is easy to verify that g is supportive; however, g is not a projection function because $g(a_1, a_2) = a_2$, whereas $g(a_1, a_3) = a_1$.

For the inductive step of the proof of Lemma 3.5, we assume that for every $n \geq 2$ and every n-ary aggregator $\bar{f} = (f_1, \ldots, m)$, there is a $d \leq n$ such that for every integer $j \leq m$ and every subset $B_j \subseteq X_j$ with at most $s - 1$ elements, the restriction $f_j \upharpoonright B_j$ is equal to pr^n_d. Fix such an n-ary aggregator $\bar{f}$ and fix an integer d, obtained by applying the induction hypothesis to $s - 1$ and $\bar{f}$. Assume, without loss of generality that $d = 1$. We will show that for every $j \leq m$ and for every subset $B_j \subseteq X_j$ of cardinality at most s, we have that $f_j \upharpoonright B_j = \mathrm{pr}^n_1$, the n-ary projection function on $d = 1$. We may assume that $s \leq n$, lest the induction hypothesis applies.

Assume towards a contradiction that there exists an integer $j_0 \leq m$ and row vectors $a^1, \ldots, a^n$ in X such that the set $B_{j_0} = \{a^1_{j_0}, \ldots, a^n_{j_0}\}$ has cardinality s and

$$f_{j_0}(a^1_{j_0}, \ldots, a^n_{j_0}) \neq a^1_{j_0}. \tag{1}$$

By supportiveness, there exists $i_0 \in \{2, \ldots, n\}$ such that

$$f_{j_0}(a^1_{j_0}, \ldots, a^n_{j_0}) = a^{i_0}_{j_0}. \tag{2}$$

Let $\{k_1, \ldots, k_s\}$ be a subset of $\{1, \ldots, n\}$ of cardinality s such that the $a^{k_1}_{j_0}, \ldots, a^{k_s}_{j_0}$ are pairwise distinct. Obviously, if $i \notin \{k_1, \ldots, k_s\}$, then there exists $l \in \{1, \ldots, s\}$ such that $a^i_{j_0} = a^{k_l}_{j_0}$. So, by renumbering we may assume that $k_1 = 1, \ldots, k_s = s$ and $i_0 = 2$. Recall that $s \geq 3$. Let $B^-_{j_0} = \{a^1_{j_0}, \ldots, a^{s-1}_{j_0}\}$. We define an $(s-1)$-ary aggregator $\bar{f}^- = (f^-_1, \ldots, f^-_m)$ as follows: first for j= $1, \ldots, m$, we set:

$$y^i_j = \begin{cases} x^i_j & \text{for } i = 1, \ldots, s-1, \\ a^s_j & \text{if } i = s, \\ a^s_j & \text{if } i > s \text{ and } a^i_{j_0} = a^s_{j_0}, \\ x^l_j & \text{for the least } l < s \text{ such that } a^i_{j_0} = a^l_{j_0}, \text{ if } i > s \text{ and } a^i_{j_0} \neq a^s_{j_0}, \end{cases} \tag{3}$$

then we set:

$$\hat{y}^i_j = \begin{cases} y^i_j & \text{if } y^i_j \neq y^2_j, \\ f_j(y^1_j, \ldots, y^n_j) & \text{otherwise}, \end{cases} \tag{4}$$

and finally we define:

$$f_j^-(x_j^1,\dots,x_j^{s-1}) = f_j(\hat{y}_j^1,\dots,\hat{y}_j^n).$$

First observe that $\bar{f}^-$ is supportive. Indeed this follows from the observation that f_j^- can never take the value a_j^s. Then observe that $\bar{f}^- = (f_1^-,\dots,f_m^-)$ is an aggregator on X, because all row vectors $y^1,\dots,y^n$ defined above belong to X (each is either some x^i or some a^i).

It is obvious that

$$f_{j_0}^-(a_{j_0}^1,\dots,a_{j_0}^{s-1}) = f_{j_0}(a_{j_0}^1,\dots,a_{j_0}^n) = a_{j_0}^2.$$

Also, let $x_{j_0}^1,\dots,x_{j_0}^{s-1} \in B_{j_0}^-$ be such that $x_{j_0}^1 \neq x_{j_0}^2$ and

$$2 \leq |\{x_{j_0}^1,\dots,x_{j_0}^{s-1}\}| \leq s-2.$$

It is easy to see that for the corresponding $\hat{y}_{j_0}^i$, it holds that

$$2 \leq |\{\hat{y}_{j_0}^1,\dots,\hat{y}_{j_0}^n\}| \leq s-1.$$

It follows that

$$f_{j_0}^-(x_{j_0}^1,\dots,x_{j_0}^{s-1}) = f_{j_0}(\hat{y}_{j_0}^1,\dots,\hat{y}_{j_0}^n) = \hat{y}_{j_0}^1 = x_{j_0}^1 \neq x_{j_0}^2.$$

Therefore, $f_{j_0}^- \restriction B_{j_0}^-$ cannot be a projection function, which contradicts the inductive hypothesis (assumed to hold for every $\bar{f}$); this concludes the proof of Lemma 3.5. □

Next, we bring into the picture some basic concepts and results from universal algebra; we refer the reader to Szendrei's monograph [14] for additional information and background. A *clone* on a finite set A is a set C of finitary operations on A (i.e., functions from a finite power of A to A) such that C contains all projection functions and is closed under arbitrary compositions (superpositions). The proof of the next lemma is straightforward.

Lemma 3.6. *Let X be a set of feasible voting patterns. For every j with $1 \leq j \leq m$ and every subset $B_j \subseteq X_j$, the set C_{B_j} of the restrictions $f_j \restriction B_j$ of the j-th components of aggregators $\bar{f} = (f_1,\dots,f_m)$ for X is a clone on B_j.*

Post [11] classified all clones on a two-element set (for more recent expositions of Post's pioneering results, see, e.g., [14] or [10]). One of Post's main findings is that if C is a clone of conservative functions on a two-element set, then either C contains only projection functions or C contains one of the following operations: the binary operation $\wedge$, the binary operation $\vee$, the ternary operation $\oplus$, the ternary operation maj.

Using all of the above, we are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. As stated earlier, only the direction $1 \implies 2$ requires proof. In the contrapositive, we will prove that if X does not admit a majority or a minority aggregator, and it does not admit a non-dictatorial binary aggregator, then X does not have an n-ary non-dictatorial aggregator, for any n. Towards this goal, and assuming that X is as stated, we will first show that the hypothesis of Lemma 3.5 holds. Once this is established, the conclusion will follow from Lemma 3.5 by taking $s = \max\{|X_j| : 1 \leq j \leq m\}$.

Given $j \leq m$ and a two-element subset $B_j \subseteq X_j$, consider the clone C_{B_j}. If C_{B_j} contained one of the binary operations $\wedge$ or $\vee$, then X would have a binary non-dictatorial aggregator, a contradiction. If, on the other hand, C_{B_j} contained the ternary operation $\oplus$ or the ternary operation maj, then, by Lemma 3.4, X would admit a minority or a majority aggregator, a contradiction as well. So, by the aforementioned Post's result, all elements of C_{B_j}, no matter what their arity is, are projection functions. By Lemma 3.4 again, since X has no binary non-dictatorial aggregator, we have that for every n and for every n-ary aggregator $\bar{f} = (f_1, \ldots, f_m)$, there exists an integer $d \leq n$ such that for every $j \leq m$ and every two-element set $B_j \subseteq X_j$, the restriction $f_j \restriction B_j$ is equal to pr^n_d, the n-ary projection on the d-th coordinate. This concludes the proof of Theorem 3.1. □

In the case of the Boolean framework, Theorem 3.1 takes the stronger form of Theorem 3.7 below. Although this result for the Boolean framework is implicit in Dokow and Holzman [7], we give an independent proof.

Theorem 3.7 (Dokow and Holzman). *Let $X \subseteq \{0,1\}^m$ be a set of feasible voting patterns. The following statements are equivalent.*

(1) X is a possibility domain.
(2) X is affine (i.e., X admits a minority aggregator) or X has a non-dictatorial binary aggregator.

Proof. Only the direction $1 \implies 2$ requires proof. Assume that X is a possibility domain in the Boolean framework. By Theorem 3.1, X admits either a majority or a minority aggregator or X has non-dictatorial binary aggregator. Since we are in the Boolean framework, this means that X is affine or X is bijunctive or X has a non-dictatorial binary aggregator. If X has at most two elements, then X is closed under $\oplus$, hence X is affine. So, it suffices to show that if X is bijunctive and has at least three elements, then X has a non-dictatorial binary aggregator. In turn, this follows immediately from the following claim.

Claim 1. *Let X be a bijunctive relation on $\{0,1\}$ with at least three elements. If X is not degenerate (i.e., every X_j has at least two elements), then X has a binary non-monomorphic aggregator.*

To prove the above claim, fix an element $\bar{a} = (a_1, \ldots, a_m) \in X$. Define the following binary aggregator, where $\bar{x} = (x_1, \ldots, x_m)$ and $\bar{y} = (y_1, \ldots, y_m)$ are arbitrary elements of X:

$$\bar{f}^{\bar{a}}(\bar{x}, \bar{y}) = (\mathrm{maj}(x_1, y_1, a_1), \ldots, \mathrm{maj}(x_m, y_m, a_m)).$$

First, observe that $\bar{f}^{\bar{a}}$ is indeed an aggregator for X. Since X is closed under maj, all we have to prove is that $\bar{f}^{\bar{a}}$ is supportive. But this is obvious, because, for $j \leq m$, if $x_j = a_j$ or $y_j = a_j$, then $\mathrm{maj}(x_j, y_j, a_j) = x_j$ or $\mathrm{maj}(x_j, y_j, a_j) = y_j$. If $x_j \neq a_j$ and $y_j \neq a_j$, then $x_j = y_j$, hence $\mathrm{maj}(x_j, y_j, a_j) = x_j = y_j$.

Now assuming that X contains more than two elements and is not degenerate, we will show that there exists a row vector $\bar{a} = (a_1, \ldots, a_m) \in X$ such that $\bar{f}^{\bar{a}}$ is not monomorphic, i.e., there are distinct $i, j = 1, \ldots, m$ such that $\bar{f}^{\bar{a}}_i \neq \bar{f}^{\bar{a}}_j$, and thus the proof of the claim will be concluded.

Observe first that if for all distinct $i \leq m$ and $j \leq m$ one of the following (depending on i, j) were true:

- for all vectors $\bar{u} \in X$, we have that $u_i = u_j$ or
- for all vectors $\bar{u} \in X$, we have that $u_i \neq u_j$,

then it would follow that there exist only two elements in X which at every coordinate have complementary values, contradicting the hypothesis that X contains more than two elements. Therefore, there exist two distinct integers $i \leq m$ and $j \leq m$ for which there are two elements $\bar{u}, \bar{v} \in X$ such that $u_i \neq u_j$ and $v_i = v_j$. Combining the last statement with the non-degeneracy of X, we conclude, by an easy case analysis, that there exist three elements $\bar{u}, \bar{v}, \bar{w} \in X$ such that at least one of the following four cases holds:

(i) the i-th and j-th coordinates of $\bar{u}, \bar{v}, \bar{w}$ are $(1,0), (0,1), (1,1)$, respectively,
(ii) the i-th and j-th coordinates of $\bar{u}, \bar{v}, \bar{w}$ are $(1,0), (0,1), (0,0)$, respectively,
(iii) the i-th and j-th coordinates of $\bar{u}, \bar{v}, \bar{w}$ are $(1,0), (1,1), (0,0)$, respectively,
(iv) the i-th and j-th coordinates of $\bar{u}, \bar{v}, \bar{w}$ are $(1,0), (0,1), (0,0)$, respectively.

In cases (i) and (ii), by computing the i-th and j-th coordinates of $\bar{f}^{\bar{u}}(\bar{u}, \bar{v})$ and $\bar{f}^{\bar{u}}(\bar{v}, \bar{u})$, we conclude that $\bar{f}_i^{\bar{u}} = \vee$ and $\bar{f}_j^{\bar{u}} = \wedge$, so $\bar{f}_i^{\bar{u}} \neq \bar{f}_j^{\bar{u}}$. In case (iii), by computing the i-th and j-th coordinates of $\bar{f}^{\bar{u}}(\bar{v}, \bar{w})$, we conclude that $\bar{f}_i^{\bar{u}} \neq \bar{f}_j^{\bar{u}}$. Case (iv) is similar. This completes the proof of Claim 1 and of Theorem 3.7. □

4 CHARACTERIZATION OF TOTAL BLOCKEDNESS

As discussed in the preceding section, much of the earlier work on possibility domains used the notion of a set being totally blocked. Our next result characterizes this notion in terms of binary aggregators and, in many respects, "explains" the role of this notion in the earlier results about possibility domains.

We begin by giving the precise definition of what it means for a set X of feasible voting patterns to be totally blocked. We will follow closely the notation and terminology used by Dokow and Holzman [8].

Let X be a set of feasible voting patterns.

- Given subsets $B_j \subseteq X_j, j = 1, \ldots, m$, the product $B = \prod_{j=1}^m B_j$ is called a *sub-box*. It is called a 2-*sub-box* if $|B_j| = 2$, for all j.

  Elements of a box B that belong also to X will be called *feasible evaluations within* B (in the sense that each issue $j = 1, \ldots, m$ is "evaluated" within B).
- Let K be a subset of $\{1, \ldots, m\}$ and let x be a tuple in $\prod_{j \in K} B_j$

  We say that x is a *feasible partial evaluation within* B if there exists a feasible evaluation y within B that extends x, i.e., $x_j = y_j$, for all $j \in K$; otherwise, we say that x is an *infeasible partial evaluation within* B.

  We say that x is a B-*Minimal Infeasible Partial Evaluation* (B-MIPE) if x is an infeasible partial evaluation within B and if for every $j \in K$, there is a $b_j \in B_j$ such that changing the j-th coordinate of x to b_j results into a feasible partial evaluation within B.
- We define a directed graph G_X as follows.

  The vertices of G_X are the pairs of *distinct* elements u, u' in X_j, for all $j = 1, \ldots m$. Each such vertex is denoted by uu'_j.

  Two vertices uu'_k, vv'_l with $k \neq l$ are connected by a directed edge from uu'_k to vv'_l if there exists a 2-sub-box $B = \prod_{j=1}^m B_j$, a set $K \subseteq \{1, \ldots, m\}$, and a B-MIPE $x = (x_j)_{j \in K}$ such that $k, l \in K$ and $B_k = \{u, u'\}$ and $B_l = \{v, v'\}$ and $x_k = u$ and $x_l = v'$. Each such directed edge is denoted by $uu'_k \underset{B,x,K}{\longrightarrow} vv'_l$ (or just $uu'_k \to vv'_l$, in case B, x, K are understood from the context).

  Notice that $uu'_k \to vv'_l$ iff $v'v_l \to u'u_k$.
- We say that X is *totally blocked* if the graph G_X is strongly connected, i.e., every two distinct vertices uu'_k, vv'_l are connected by a directed path (this must hold even if $k = l$). This notion, defined in Dokow and Holzman [8], is a generalization to the case where the A_j's are allowed to have arbitrary cardinalities of a corresponding notion for the Boolean framework (every A_j has cardinality 2), originally given in [9].

We are now ready to state the following result.

Theorem 4.1. *Let X be a set of feasible voting patterns. The following statements are equivalent.*

(1) X is totally blocked.
(2) X has no non-dictatorial binary aggregator.

Observe that Theorem 3.7 is also an immediate consequence of Theorem A and Theorem 4.1. In view of Theorem B by Dokow and Holzman [8], only the direction $1 \implies 2$ of Theorem 4.1 requires proof. Nevertheless, we prove both directions of Theorem 4.1 for completeness.

Proof. We start with direction $1 \implies 2$. Consider at first two vertices uu'_k, vv'_l of G_X (with $k \neq l$) connected by an edge $uu'_k \to vv'_l$. Then there exists a 2-sub-box $B = \prod_{j=1}^{m} B_j$ with $B_k = \{u, u'\}$ and $B_l = \{v, v'\}$ and a B-MIPE $x = (x_j)_{j \in K}$ such that $\{k, l\} \subseteq K$ and $x_k = u, x_l = v'$.

Claim 2. *For every binary aggregator $\bar{f} = (f_1, \ldots, f_m)$ of X, if $f_k(u, u') = u$, then $f_l(v, v') = v$.*

Proof of Claim. By the minimality of x within B if we flip x_k from u to u' or if we flip x_l from v' to v, then we get, in both cases, respective feasible evaluations within B. Therefore, there are two total evaluations e and e' in $X \cap B$ such that

- $e_k = u'$ and
- $e_s = x_s$ for $s \in K, s \neq k$ (in particular $e_l = v'$),

and

- $e'_l = v$ and
- $e'_s = x_s$ for $s \in K, s \neq l$ (in particular $e'_k = u$).

If we assume, towards a contradiction, that $f_k(u, u') = u$ and $f_l(v, v') = v'$, we immediately have that the evaluation

$$\bar{f}(e, e') := (f_1(e_1, e'_1), \ldots, f_m(e_m, e'_m))$$

extends $(x_j)_{j \in K}$, contradicting the latter's infeasibility within B. This completes the proof of the Claim and we now return to the proof of Theorem 4.1.

From the Claim we get that if $uu'_k \to\to vv'_l$ and $f_k(u, u') = u$, then $f_l(v, v') = v$ (even if $k = l$), where $uu'_k \to\to vv'_l$ means that there is path from uu'_k to vv'_l in the graph G_X. Also, since by supportiveness $f_l(v, v') \in \{v, v'\}$, we have that if $vv'_l \to\to uu'_k$ and $f_k(u, u') = u'$, then $f_l(v, v') = v'$. From this, it immediately follows that if G_X is strongly connected, then every binary aggregator of X is dictatorial.

We will now prove Direction $2 \implies 1$ of Theorem 4.1 , namely, that if X is not totally blocked, then there is a non-dictatorial binary aggregator (this part is contained in [8, Theorem 2] –Theorem B above). Since G_X is not strongly connected, there is a partition of the vertices of G_X into two mutually disjoint and non-empty subsets V_1 and V_2 so that there is no edge from a vertex of V_1 towards a vertex in V_2. We now define a $\bar{f} = (f_1, \ldots, f_m)$, where $f_k : A_k^2 \mapsto A_k$, as follows:

$$f_k(u, u') = \begin{cases} u & \text{if } u, u' \in X_k \text{ and } uu'_k \in V_1 \text{ and } u \neq u', \\ u' & \text{if } u, u' \in X_k \text{ and if } uu'_k \in V_2 \text{ and } u \neq u', \\ u & \text{if } u = u' \text{ or } u \in A_k \setminus X_k \text{ or } u' \in A_k \setminus X_k. \end{cases} \tag{5}$$

In other words, for two differing values u and u' in X_k, the function f_k is defined as the projection on the first coordinate if $uu'_k \in V_1$, and as the projection onto the second coordinate if $uu'_k \in V_2$; we also define $f_k(u, u) = u$ if $u = u'$ or if either u or u' is not in X_k (i.e., when at least one of them is not a projection onto the k-th coordinate of an element of X, in this latter case the value of $f_k(u, u')$ can be arbitrarily defined, as it has no effect on the properties of $\bar{f}$).

Notice that $\bar{f}$ is non-dictatorial, because V_1 and V_2 are not empty.

All that remains to be shown is that X is closed under $\bar{f}$, i.e., if $e = (e_1, \ldots, e_m), e' = (e'_1, \ldots, e'_m) \in X$ are two total feasible evaluations, then

$$\bar{f}(e, e') := (f_1(e_1, e'_1), \ldots f_m(e_m, e'_m)) \in X. \tag{6}$$

Let

$$L = \{j = 1, \ldots, m \mid e_j \neq e'_j\}.$$

For an arbitrary $j \in L$, define vertex$_j(e, e')$ to be the vertex uu'_j of G_X, where $u = e_j$ and $u' = e'_j$.

If now $\bar{f}(e, e') = e$ or if $\bar{f}(e, e') = e'$, then obviously (6) is satisfied. So assume that

$$\bar{f}(e, e') \neq e \text{ and } \bar{f}(e, e') \neq e'. \tag{7}$$

Also, towards showing (6) by contradiction, assume

$$\bar{f}(e, e') \notin X. \tag{8}$$

Define now a 2-sub-box $B = (B_j)_{j=1,\ldots,m}$ as follows:

$$B_j = \begin{cases} \{e_j, e'_j\} & \text{if } e_j \neq e'_j, \\ \{e_j, a_j\} & \text{otherwise}, \end{cases} \tag{9}$$

where a_j is an arbitrary element $\neq e_j$ of X_j (the latter choice is only made to ensure that $|B_j| = 2$ in all cases).

Because of (8) and (9), we have that $\bar{f}(e, e')$ is a total evaluation infeasible within B. Towards constructing a B-MIPE, delete one after the other (and as far as it can go) coordinates of $\bar{f}(e, e')$, while taking care not to destroy infeasibility within B. Let $K \subseteq \{1, \ldots, m\}$ be the subset of coordinate indices that remain at the end of this process. Then the partial evaluation

$$x := \left(f_j(e_j, e'_j)\right)_{j \in K} \tag{10}$$

is infeasible within B. Therefore, lest e or e' extends $x = \left(f_j(e_j, e'_j)\right)_{j \in K}$ (not permissible because the latter partial evaluation is infeasible), there exist $k, l \in K$ such that

$$e_k \neq e'_k \text{ and } e_l \neq e'_l \tag{11}$$

and also

$$f_k(e_k, e'_k) = e_k \text{ and } f_l(e_l, e'_l) = e'_l. \tag{12}$$

But then if we set

$$u = e_k, u' = e'_k, v = e_l, v' = e'_l, \tag{13}$$

we have, by (5), (11), (12) and (13), that

$$\text{vertex}_k(e, e') = uu'_k \in V_1 \text{ and } \text{vertex}_l(e, e') = vv'_l \in V_2 \tag{14}$$

and, by (5), (12) and (13), we get that

$$uu'_k \underset{B,x,K}{\longrightarrow} vv'_l$$

which by (14) is a contradiction, because we get an edge from V_1 to V_2. This completes the proof of Theorem 4.1. ◻

Before proceeding further, we point out that the three types of non-dictatorial aggregators in Theorem 3.1 are, in a precise sense, independent of each other.

Example 4.2. Consider the set $X = \{0, 1\}^3 \setminus \{(1, 1, 0)\}$ of satisfying assignments of the Horn clause $(\neg x \vee \neg y \vee z)$.

It is easy to see that X is closed under the binary operation $\wedge$, but it is not closed under the ternary majority operation maj or the ternary minority operation $\oplus$.

Thus, X is a possibility domain admitting a non-dictatorial binary aggregator, but not a majority aggregator or a minority aggregator. ◻

Example 4.3. Consider the set $X = \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$ of solutions of the equation $x + y + z = 1$ over the two-element field.

It is easy to see that X is closed under the ternary minority operation $\oplus$, but it is not closed under the ternary majority operation maj. Moreover, Dokow and Holzman [7, Example 3] pointed out that X is totally blocked, hence Theorem 4.1 implies that X does not admit a non-dictatorial binary aggregator.

Thus, X is a possibility domain admitting a minority aggregator, but not a majority aggregator or a non-dictatorial binary aggregator. □

Example 4.4. Consider the set $X = \{(0,1,2),(1,2,0),(2,0,1),(0,0,0)\}$.

This set was studied in [8, Example 4]. It can be shown that X admits a majority aggregator. To see this, consider the ternary operator $\overline{f} = (f_1, f_2, f_3)$ such that $f_j(x,y,z)$ is the majority of x, y, z, if at least two of the three values are equal, or it is 0 otherwise. Notice that in the latter case the value 0 must be one of the x, y, z, so this operator is indeed supportive. It is easy to verify that X is closed under (f_1, f_2, f_3). Moreover, if one of the f_j's is restricted to a two-element domain (i.e., to one of $\{0,1\}$, $\{(1,2)\}$, $\{0,2\}$), then it must be the majority function by its definition, so $\overline{f}$ is indeed a majority aggregator on X.

Dokow and Holzman argued that X is totally blocked, hence Theorem 4.1 implies that X does not admit a non-dictatorial binary aggregator.

Next, we claim that X does not admit a minority aggregator. Towards a contradiction, assume it admits the minority aggregator $\overline{g} = (g_1, g_2, g_3)$. By applying $\overline{g}$ to the triples $(0,1,2)$, $(1,2,0)$, $(0,0,0)$ in X, we infer that the triple $(g_1(0,1,0), (g_2(1,2,0), g_3(2,0,0))$ must be in X. By the assumption that this aggregator is the minority operator on two-element domains, we have that $g_1(0,1,0) = 1$ and $g_3(2,0,0) = 2$, so X contains a triple of the form $(1, g_2(1,2,0), 2)$; however, X contains no triple whose first coordinate is 1 and its third coordinate is 2, so we have arrived at a contradiction.

Thus, X is a possibility domain admitting a majority aggregator, but not a minority aggregator or a non-dictatorial binary aggregator. □

Observe that the possibility domains in Examples 4.2 and 4.3 are in the Boolean framework, while the possibility domain in Example 4.4 is not. This is no accident, because it turns out that, in the Boolean framework, if a set admits a majority aggregator, then it also admits a non-dictatorial binary aggregator. This property is shown as a Claim in the proof of Theorem 3.7. Note also that this explains why admitting a majority aggregator is not part of the characterization of possibility domains in the Boolean framework in Theorem 3.7.

5 UNIFORM POSSIBILITY DOMAINS

In this Section, we connect aggregation theory with multi-sorted constraint satisfaction problems. Towards this goal, we introduce the following stronger notion of a non-dictatorial aggregator.

Definition 5.1. Let X be a set of feasible voting patterns.

- We say that an aggregator $\overline{f} = (f_1, \ldots, f_m)$ for X is *uniform non-dictatorial* if for every $j = 1, \ldots, m$ and every two-element subset $B_j \subseteq X_j$, we have that $f_j \restriction B_j$ is not a projection function.
- We say that X is a *uniform possibility domain* if X admits a uniform non-dictatorial aggregator of some arity.

The next example shows that the notion of a uniform possibility domain is stricter than the notion of a possibility domain.

Example 5.2. Let $W = \{(1,0,0),(0,1,0),(0,0,1)\}$ be the 1-in-3 relation, considered in Example 2.6. As seen earlier, the cartesian product $W \times W$ is a possibility domain. We claim that $W \times W$ is not a uniform possibility domain in the sense of Definition 5.1. Indeed, since W is an impossibility domain, it follows easily that for every n, all n-ary aggregators of $W \times W$ are of the form

$$(pr^n_d, pr^n_d, pr^n_d, pr^n_{d'}, pr^n_{d'}, pr^n_{d'}), \text{ for } d, d' \in \{1, \dots, n\}. \qquad \square$$

It is obvious that every set X that admits a majority aggregator or a minority aggregator is a uniform possibility domain. The next example states that uniform possibility domains are closed under cartesian products.

Example 5.3. If X and Y are uniform possibility domains, then so is their cartesian product $X \times Y$.

Assume that $X \subseteq \prod_{j=1}^{l} A_j$ and $Z \subseteq \prod_{j=l+1}^{m} A_j$, where $1 \leq l < m$. Let $(f_1, \dots, f_l)$ be a uniform non-dictatorial aggregator for X and let $(f_{l+1} \dots, f_m)$ be a uniform non-dictatorial aggregator for X. Then

$$(f_1, \dots, f_l, f_{l+1}, \dots, f_m)$$

is a uniform non-dictatorial aggregator for $X \times Y$. $\square$

Let B be an arbitrary two-element set, viewed as the set $\{0,1\}$, and consider the binary logical operations $\wedge$ and $\vee$ on B (since we will always deal with both these logical operations concurrently, it does not matter which element of B we take as 0 and which as 1). For notational convenience. we define two ternary operations on B as follows:

$$\wedge^{(3)}(x,y,z) = x \wedge y \wedge z \;\text{ and }\; \vee^{(3)}(x,y,z) = x \vee y \vee z.$$

We now first state and subsequently prove the following result.

THEOREM 5.4. *Let X be a set of feasible voting patterns. The following statements are equivalent.*

(1) X is a uniform possibility domain.
(2) For every $j = 1, \dots, m$ and for every two-element subset $B_j \subseteq X_j$, there is an aggregator $\bar{f} = (f_1, \dots, f_m)$ (that depends on j and B_j) of some arity such that $f_j \restriction B_j$ is not a projection function.
(3) There is a ternary aggregator $\bar{f} = (f_1, \dots, f_m)$ such that for all $j = 1, \dots, m$ and all two-element subsets $B_j \subseteq X_j$, we have that $f_j \restriction B_j$ is one of the ternary operations $\wedge^{(3)}$, $\vee^{(3)}$, maj, $\oplus$ (to which of these four ternary operations the restriction $f_j \restriction B_j$ is equal to depends on j and B_j).
(4) There is a ternary aggregator $\bar{f} = (f_1, \dots, f_m)$ such that for all $j = 1, \dots, m$ and all $x, y \in X_j$, we have that $f_j(x,y,y) = f_j(y,x,y) = f_j(y,y,x)$.

Before the proof of Theorem 5.4, we give several preliminaries.
We start with the following lemma:

LEMMA 5.5 (SUPERPOSITION OF AGGREGATORS). *Let $\bar{f} = (f_1, \dots, f_m)$ be an n-ary aggregator and let*

$$\overline{h^1} = (h^1_1, \dots, h^1_m), \dots, \overline{h^n} = (h^n_1, \dots, h^n_m)$$

be n k-ary aggregators (all on m issues). Then the m-tuple of k-ary functions $(g_1, \dots, g_m)$ defined by:

$$g_j(x_1, \dots, x_k) = f_j(h^1_j(x_1, \dots, x_k), \dots, h^n_j(x_1, \dots, x_k)), j = 1, \dots, m$$

is also an aggregator.

Proof. Let $x^l_j, l = 1, \ldots, k, j = 1, \ldots, m$ be a $k \times m$ matrix whose rows are in X. Since the $\overline{h^i}, i = 1, \ldots, n$ are k-ary aggregators, we conclude that for all $i = 1, \ldots, n$,

$$(h^i_1(x^1_1, \ldots, x^k_1), \ldots, h^i_m(x^1_m, \ldots, x^k_m)) \in X.$$

We now apply the aggregator $\bar{f} = (f_1, \ldots, f_m)$ to the $n \times m$ matrix

$$h^i_j(x^1_j, \ldots, x^k_j), i = 1, \ldots, n, j = 1, \ldots, m,$$

which concludes the proof. □

Using the above lemma we will assume below, often tacitly, that various tuples of functions obtained by superposition of aggregators with other aggregators, like projections, are aggregators as well.

We now prove three lemmas:

Lemma 5.6. *Let A be an arbitrary set and $f : A^3 \mapsto A$ a ternary supportive operation on A, and B a two-element subset of A taken as $\{0, 1\}$. Then $f{\restriction}B$ is commutative iff $f{\restriction}B \in \{\wedge^{(3)}, \vee^{(3)}, \text{maj}, \oplus\}$.*

Proof. Only the sufficiency of commutativity of $f{\restriction}B$ for its being one of $\wedge^{(3)}, \vee^{(3)}$, maj, $\oplus$ is not entirely trivial. Since f is supportive, $f(0, 0, 0) = 0$ and $f(1, 1, 1) = 1$. Assume $f{\restriction}\{0, 1\}$ is commutative. Let

$$f(1, 0, 0) = f(0, 1, 0) = f(0, 0, 1) := a, \text{ and}$$
$$f(0, 1, 1) = f(1, 0, 1) = f(1, 1, 0) := b.$$

By supportiveness, $a, b \in \{0, 1\}$. If $a = b = 0$, then $f = \wedge^{(3)}$; if $a = b = 1$, $f = \vee^{(3)}$; if $a = 0$ and $b = 1$, $f =$ maj; and if $a = 1$ and $b = 0$, $f = \oplus$. □

Lemma 5.7. *Let A be an arbitrary set and $f, g : A^3 \mapsto A$ two ternary supportive operations on A. Define the supportive as well ternary operation*

$$h(x, y, z) = f(g(x, y, z), g(y, z, x), g(z, x, y)).$$

If B is a two-element subset of A then $h{\restriction}B$ is commutative if either $f{\restriction}B$ or $g{\restriction}B$ is commutative.

Proof. The result is entirely trivial if $g{\restriction}B$ is commutative, since in this case, by supportiveness of f, $h{\restriction}B = g{\restriction}B$. If on the other hand $f{\restriction}B$ is commutative then easily from the definition of h follows that for any $x, y, z \in B$, $h(x, y, z) = h(y, z, x) = h(z, x, y)$. This form of superposition of f and g appears also in Bulatov [4, Section 4.3]. □

For notational convenience, we introduce the following definition:

Definition 5.8. Let $\bar{f}$ and $\bar{g}$ be two aggregators on X. Let $\bar{f} \diamond \bar{g}$ be the ternary aggregator $\bar{h} = (h_1, \ldots, h_m)$ defined by:

$$h_j(x, y, z) = f_j(g_j(x, y, z), g_j(y, z, x), g_j(z, x, y)), j = 1, \ldots, m,$$

(The fact that $\bar{h}$ is indeed an aggregator follows from Lemma 5.5 and the fact that a tuple of functions comprised of the same projections is an aggregator.)

Lemma 5.9. *Let $\bar{f}$ and $\bar{g}$ be two aggregators on X. Let $i, j \in \{1, \ldots, m\}$ two arbitrary issues (perhaps identical) and B_i, B_j two two-element subsets of X_i and X_j, respectively. If $f_i{\restriction}B_i$ and $g_j{\restriction}B_j$ are commutative (i.e., by Lemma 5.6 if each is one of the $\wedge^{(3)}, \vee^{(3)}$, maj, $\oplus$) then both $\bar{f} \diamond \bar{g}{\restriction}B_i$ and $\bar{f} \diamond \bar{g}{\restriction}B_j$ are commutative (i.e., each is one of the $\wedge^{(3)}, \vee^{(3)}$, maj, $\oplus$).*

Proof. Immediate by Lemmas 5.6 and 5.7. □

We now prove the characterization of uniform possibility domains. Some of the techniques employed in the proof of Theorem 5.4 and the preceding lemmas had been used by Bulatov (see [3, Proposition 3.1] [4, Proposition 2.2]; these results however consider only operations of arity two or three.)[9]

Proof of Theorem 5.4.

The directions (1) $\implies$ (2) and (3) $\implies$ (1) are obvious. Also the equivalence of (3) and (4) immediately follows from Lemma 5.6. It remains to show (2) $\implies$ (3). For a two-element subset $B_j \subseteq X_j$, let C_{B_j} be the clone (Lemma 3.6) of the restrictions $f_j \restriction B_j$ of the j-th components of aggregators $\bar{f} = (f_1, \ldots, f_m)$. By Post [11], we can easily get that C_{B_j} contains one of the operations $\wedge, \vee$, maj and $\oplus$. Therefore, easily, for all j, B_j there is a ternary aggregator $\bar{f} = (f_1, \ldots, f_m)$ (depending on j, B_j) such that $f_j \restriction B_j$ is one of the $\wedge^{(3)}, \vee^{(3)}$, maj and $\oplus$. Now let $\bar{f}^1, \ldots, \bar{f}^N$ be an arbitrary enumeration of all ternary aggregators each of which on some issue j and some two-element B_j is one of the $\wedge^{(3)}, \vee^{(3)}$, maj and $\oplus$ and such that the $\bar{f}^l$'s cover all possibilities for j, B_j. As a ternary operation $\bar{h}$ such that uniformly for each j, B_j, the restriction $h_j \restriction B_j$ belongs to the set $\{\wedge^{(3)}, \vee^{(3)}, \text{maj}, \oplus\}$ we can take, by Lemma 5.9,

$$(\cdots(\bar{f}^1 \diamond \bar{f}^2) \diamond \cdots \diamond \bar{f}^N),$$

which concludes the proof. □

To state our result that connects the property of X being a uniform possibility domain with the property of tractability of a multi-sorted constraint satisfaction problems, we first introduce some notions following closely [5] and [3].

As before, we consider a fixed set $I = \{1,, \ldots, m\}$, but this time I represents *sorts*. We also consider a family $\mathcal{A} = \{A_1, \ldots, A_m\}$ of finite sets, each of cardinality at least 2, representing the values the corresponding sorts can take.

- Let $(i_1, \ldots, i_k)$ be a list of (not necessarily distinct) indices from I. A *multi-sorted relation* over $\mathcal{A}$ with arity k and signature $(i_1, \ldots, i_k)$ is a subset R of $A_{i_1} \times \cdots \times A_{i_k}$, together with the list $(i_1, \ldots, i_k)$. The signature of such a multi-sorted language R will be denoted $\sigma(R)$.
- A *multi-sorted constraint language* Γ over $\mathcal{A}$ is a set of multi-sorted relations over $\mathcal{A}$.

Definition 5.10 (Multi-sorted CSP). Let Γ be a multi-sorted constraint language over a family $\mathcal{A} = \{A_1, \ldots, A_m\}$ of finite sets. The multi-sorted constraint satisfaction problem MCSP(Γ) is the following decision problem.

An instance of MCSP(Γ) is a quadruple $(V, \mathcal{A}, \delta, C)$, where V is a finite set of variables; δ is a mapping from V to I, called the sort-assignment function (v belongs to the sort $\delta(v)$); C is a set of constraints where each constraint $C \in C$ is a pair (s, R), such that $s = (v_1, \ldots, v_k)$ is a tuple of variables of length k, called the constraint scope; R is a k-ary multi-sorted relation over $\mathcal{A}$ with signature $(\delta(v_1), \ldots, \delta(v_k))$, called the constraint relation.

The question is whether a value-assignment exists, i.e., a mapping $\phi : V \mapsto \bigcup_{i=1}^m A_i$, such that, for each variable $v \in V$, we have that $\phi(v) \in A_{\delta(v)}$, and for each constraint $(s, R) \in C$, with $s = (v_1, \ldots, v_k)$, we have that the tuple $(\phi(v_1), \ldots, \phi(v_k))$ belongs to R.

A multi-sorted constraint language Γ over $\mathcal{A}$ is called *conservative* if for all sets $A_j \in \mathcal{A}$ and all subsets $B \subseteq A_j$, we have that $B \in \Gamma$ (as a relation over A_j).

If $X \subseteq \prod_{j=1}^m A_j$ is a set of feasible voting patterns, then X can be considered as multi-sorted relation with signature $(1, \ldots, m)$ (one sort for each issue). We write Γ_X^{cons} to denote the multi-sorted conservative constraint language consisting of X and all subsets of every $A_j, j = 1, \ldots, m$, the latter considered as relations over A_j.

[9] This came to the attention of the authors only after the work reported here had been essentially completed.

If the sets A_j are equal to each other and $|I| = 1$, i.e., if there is no differentiation between sorts, then $\mathrm{MCSP}(\Gamma)$ is denoted the constraint satisfaction problem $\mathrm{CSP}(\Gamma)$. If the sets of votes for all issues are equal, then it is possible to consider a feasible set of votes X as a one-sorted relation (all issues are of the same sort). In this framework, and in case all A_j's are equal to $\{0, 1\}$, we have that $\mathrm{CSP}(\Gamma_X^{\mathrm{cons}})$ coincides with the problem introduced by Schaefer [12], which he called the "generalized satisfiability problem with constants" and denoted by $\mathrm{SAT_C}(\{X\})$. Note that the presence of the sets $\{0\}$ and $\{1\}$ in the constraint language amounts to allowing constants, besides variables, in the constraints.

Schaefer [12] proved a prototypical dichotomy theorem for the complexity of the generalized satisfiability problem with constants. Bulatov [3, Theorem 2.16] proved a dichotomy theorem for conservative multi-sorted constraint languages, which in our setting reads:

Dichotomy Theorem (Bulatov). If for any $j = 1, \ldots, m$ and any two-element subset $B_j \subseteq X_j$ there is either a binary aggregator $\bar{f} = (f_1, \ldots, f_m)$ such that $\bar{f}_j \restriction B_j \in \{\wedge, \vee\}$ or a ternary aggregator $\bar{f} = (f_1, \ldots, f_m)$ such that $\bar{f}_j \restriction B_j \in \{\mathrm{maj}, \oplus\}$, then $\mathrm{MCSP}(\Gamma_X^{\mathrm{cons}})$ is solvable in polynomial time; otherwise it is NP-complete.

We now state the following dichotomy theorem.

Theorem 5.11. *If X is a uniform possibility domain, then* $\mathrm{MCSP}(\Gamma_X^{\mathrm{cons}})$ *is solvable in polynomial time; otherwise it is NP-complete.*

Proof. The tractability part of the statement follows from Bulatov's Dichotomy Theorem and item (3) of Theorem 5.4 (observing that $x \wedge y = \wedge^{(3)}(x, x, y)$ and similarly for $\vee$ and using Lemma 5.5), whereas the completeness part follows from Bulatov's Dichotomy Theorem and item (2) of Theorem 5.4. □

We end this section with the following example:

Example 5.12. Let $Y = \{0, 1\}^3 \setminus \{(1, 1, 0)\}$ be the set of satisfying assignments of the clause $(\neg x \vee \neg y \vee z)$ and let $Z = \{(1, 1, 0), (0, 1, 1), (1, 0, 1), (0, 0, 0)\}$ be the set of solutions of the equation $x + y + z = 0$ over the two-element field.

We claim that Y and Z are uniform possibility domains, hence, by Example 5.3, the cartesian product $X = Y \times Z$ is also a uniform possibility domain. From Theorem 5.11, it follows that $\mathrm{MCSP}(\Gamma_X^{\mathrm{cons}})$ is solvable in polynomial time. However, the generalized satisfiability problem with constants $\mathrm{SAT_C}(\{X\})$ (equivalently $\mathrm{CSP}(\Gamma_X^{\mathrm{cons}})$) is NP-complete.

Indeed, in Schaefer's [12] terminology, the set Y is Horn (equivalently, it is coordinate-wise closed under $\wedge$); however, it is not dual Horn (equivalently, it is not coordinate-wise closed under $\vee$), nor affine (equivalently, it does not admit a minority aggregator) nor bijunctive (equivalently, it does not admit a majority aggregator). Therefore, by coordinate-wise closure under $\wedge$, we have that Y is a uniform possibility domain. Also, Z is affine, but not Horn, nor dual Horn neither bijunctive. So, being affine, Z is a uniform possibility domain. The NP-completeness of $\mathrm{SAT_C}(\{X\})$ (equivalently, the NP-completeness of $\mathrm{CSP}(\Gamma_X^{\mathrm{cons}})$) follows from Schaefer's dichotomy theorem [12], because X is not Horn, dual Horn, affine, nor bijunctive. □

6 CONCLUDING REMARKS

In this paper, we used algebraic tools to investigate the structural properties of possibility domains, that is, domains that admit non-dictatorial aggregators. We also established a connection between the stronger notion of a uniform possibility domain and multi-sorted constraint satisfaction. We conclude by discussing two algorithmic problems that underlie the notions of a possibility domain and a uniform possibility domain.

Given a family $\mathcal{A} = \{A_1, \ldots, A_m\}$ and a subset $X \subseteq \prod_{j=1}^m A_j$ as input, adopting a terminology used in computational complexity theory, we call *meta-problems* the following two questions:

(i) Is X a possibility domain?
(ii) Is X a uniform possibility domain?

Theorem 3.1 (in fact, even Corollary 2.7) and, respectively, Theorem 5.4, easily imply that the meta-problem (i) and, respectively, the meta-problem (ii), is in NP. Indeed, we only have to guess suitable ternary or binary operations and check for closure. However, even if the sizes of all A_j's are bounded by a constant (but the number m of issues/sorts, is unbounded), it is conceivable that the problems are not in polynomial time, as there are exponentially many ternary or binary aggregators. The question of pinpointing the exact complexity of these two meta-problems is the object of ongoing research. Of course, if, besides the cardinality of all sets A_j, their number m is also bounded, then Theorem 3.1 (in fact, even Corollary 2.7) and respectively, Theorem 5.4 imply that the meta-problem (i) and, respectively, the meta-problem (ii)) is solvable in polynomial time (for the first meta-problem, this was essentially observed by Szegedy and Xu [13]). Note that, in the preceding considerations, it is assumed that X is given by listing explicitly its elements. If X is given implicitly in a succinct way (e.g., as the set of satisfying assignments of a given Boolean formula), then the upper bound for the meta-problems is higher. The exact complexity of the aforementioned meta-problems with X represented succinctly remains to be investigated.

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