Date

Presentation for 20 April 2021

Social Choice Theory

Berwin Gan

Main Result

Only Borda count - U,P,ND, MIIA,A,N and PR

Proof

When $|X| = 2$, May's theorem = Borda count.

Let $X = \{x, y, z\}$ for $|X| = 3$
For profile $\succ_.$
$a_{xy}(\succ_.)$ – $(x \succ y \succ z)$ or $(z \succ x \succ y)$
$a_{yx}(\succ_.)$ – $(z \succ y \succ x)$ or $(y \succ x \succ z)$

For $I_3^F(a_{xy}, a_{yx})$ be proportion a_{xzy}
If $a_{yzx}(\succ_.) = 1 - a_{xy} - a_{yx} - I_3^F(a_{xy}, a_{yx})$, then $x \sim_F y$, where $\succeq_F = F(\succ_.)$
I_3^F Social Indifference Curve

Logic Pause

Case 1: if $a_{xy} = a_{yx}$, then $a_{xzy} = a_{yzx}$ if $x \sim y$

Case 2: if $a_{xy} = 1$, $I_3^F(a_{xy}, a_{yx}) = 0$

Case 3: if $a_{xy} = a_{yx} = 0$, then $a_{xzy} = a_{yzx}$

Continue Proof

Fractions does not affect the social ranking of x and y

$a_{xy}(\succ_{.})$ – $(x \succ y \succ z)$ or $(z \succ x \succ y)$

Actual division does not matter

Continue Proof

$I_3^F(a_{xy}, a_{yx})$ is unique.

Two values: a_{xzy} and a'_{xzy} with $a_{xzy} < a'_{xzy}$

From $(a_{xy}, a_{yz}, a_{xzy}, 1 - a_{xy} - a_{yz} - a_{xzy})$ to $(a_{xy}, a_{yz}, a'_{xzy}, 1 - a_{xy} - a_{yz} - a'_{xzy})$ means x is rising and y is falling in individual rankings.

$x \sim_F y$ for both is not possible.

Continue Proof

Show that $I_3^F = I_3^B$ in order for F to be the Borda Count rule.

Category 1				Category 2	
x	z	y	z	x	y
y	x	x	y	z	z
z	y	z	x	y	x

Sensitivity between category 1 and 2

For $x \sim y$, $B(x) = B(y)$

$0 = (a_{xy} - a_{yx}) + 2(a_{xzy} - a_{yzx})$ **(eqn 1)**
$0 = (a_{xy} - a_{yx}) + 2(I_3^B - (1 - a_{xy} - a_{yx} - I_3^B))$ **(eqn 2)**
$0 = (a_{xy} - a_{yx}) + 2(2I_3^B - 1 + a_{xy} + a_{yx})$
$0 = a_{xy} - a_{yx} + 4I_3^B - 2 + 2a_{xy} + 2a_{yx})$
$0 = 4I_3^B - 2 + 3a_{xy} + a_{yx})$

$I_3^B = (2 - 3a_{xy} - a_{yx})/4$ **(eqn 3)**

Clarification: I_3^B shortform for $I_3^B(a_{xy}, a_{yx})$

Assumption for the general form
$I_3^F(a_{xy}, a_{yx}) = B_0 + B_{xy}a_{xy} + B_{yx}a_{yx}$ for some B_0, B_{xy}, B_{yx}. **(eqn 4)**

When $a_{xy} = a_{yx} = a$ from case 1
$I_3^F(a, a) = 1 - 2a - I_3^F(a, a)$ **(eqn 5)**

$2I_3^F(a, a) = 1 - 2a = 2B_0 + 2(B_{xy} + Byx)a$ **(eqn 6)**

Infer:
$B_0 = \frac{1}{2}$
$B_{xy} + B_{yx} = -1$

Consider the profile $\succ^*$.

$\frac{1}{6}+c$	$\frac{1}{6}+c$	$\frac{1}{6}+c$	$\frac{1}{6}-c$	$\frac{1}{6}-c$	$\frac{1}{6}-c$
x	z	y	z	y	x
y	x	z	y	x	z
z	y	x	x	z	y

Claim: $x \sim y$

Counter-claim: $x \succ^*_F y$

Permutation σ: $\sigma(x) = y$,$\sigma(y) = z$ and $\sigma(z) = x$

$\frac{1}{6}+c$	$\frac{1}{6}+c$	$\frac{1}{6}+c$	$\frac{1}{6}-c$	$\frac{1}{6}-c$	$\frac{1}{6}-c$
y	x	z	x	z	y
z	y	x	z	y	x
x	z	y	y	x	z

Claim for second profile: $y \succ^*_F z$

Contradiction:$x \succ^*_F y \succ^*_F z \succ^*_F x$

$\frac{1}{6}+c$	$\frac{1}{6}+c$	$\frac{1}{6}+c$	$\frac{1}{6}-c$	$\frac{1}{6}-c$	$\frac{1}{6}-c$
x	z	y	z	y	x
y	x	z	y	x	z
z	y	x	x	z	y

$a_{xy} = \frac{1}{3} + 2c$
$a_{yx} = \frac{1}{3} - 2c$
$a_{xzy} = \frac{1}{6} - c$

$I_3^F(\frac{1}{3} + 2c, \frac{1}{3} - 2c) = \frac{1}{6} - c$ (eqn 7)

CONTINUE PROOF

$I_3^F(\frac{1}{3}+2c, \frac{1}{3}-2c) = \frac{1}{6} - c$ **(eqn 7)**
$I_3^F(a_{xy}, a_{yx}) = B_0 + B_{xy}a_{xy} + B_{yx}a_{yx}$ for some B_0, B_{xy}, B_{yx}. **(eqn 4)**
$B_0 = \frac{1}{2}$
$B_{xy} + B_{yx} = -1$

$\frac{1}{2} + B_{xy}(\frac{1}{3}+2c) - (1+B_{xy})(\frac{1}{3}-2c) = \frac{1}{6} - c$ **(eqn 8)**

$B_{xy} = -\frac{3}{4}$
$B_{yx} = -\frac{1}{4}$