Presentation for 17 March 2021

Social Choice Theory

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Theorem 3.1

Theorem

Let X be a set of feasible voting patterns.

- *X is a possibility domain.*
- *X admits non-dictatorial binary aggregator or a majority aggregator or minority aggregator.*

Note on Binary Aggregator

When restricted to two-element subset.

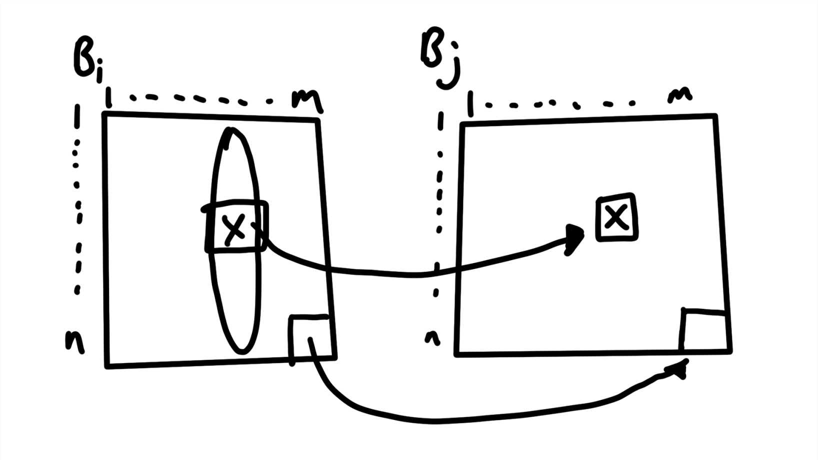
- projection
- $\wedge$ function
- $\vee$ function

Definition

A function $\overline{f}$ is monomorphic if for all $1 \leq i, j \leq m$ and for all two-element subset $B_i \subseteq X_i$ and $B_j \subseteq X_i$ and every bijection $g : B_i \mapsto B_j$ and all column vector $x_i = (x_i^1, ..., x_i^n) \in B_i^n$

$$f_j(g(x_i^1), ..., g(x_i^n) = g(f_i(x_i^1, ..., x_i^n))$$

Visual Aid



Example 3.3

Let X be a set of feasible voting patterns that admits a minority or majority ternary aggregator $\bar{f}$. Then $\bar{f}$ is locally monomorphic.

Proof.

Let $\bar{f}$ be a minority ternary aggregator. For every $1 \leq i, j \leq m$, let $B_i = \{a, b\} \subseteq X_i$ and $B_j = \{c, d\} \subseteq X_i$

$$g(a) = c$$
$$g(b) = d$$
$$g'(a) = d$$
$$g'(b) = c$$

L

Proof.

Let (x, y, z) be a triple in $x, y, z \in B_i$.
Without loss of generality, let $x = a, y = z = b$.

$$
\begin{aligned}
& f_j(g(x), g(y), g(z)) \\
\Leftrightarrow\ & f_j(c, d, d) \\
\Leftrightarrow\ & \bigoplus(c, d, d) \\
\Leftrightarrow\ & c \\
\Leftrightarrow\ & g(a) = g(\bigoplus(a, b, b)) \\
\Leftrightarrow\ & g(f_i(x, y, z))
\end{aligned}
$$

The same holds for g'. Since i, j were arbitrary, $\bar{f}$ is locally monomorphic. $\square$

Lemma 3.4

Let X be the set of feasible voting patterns. If every binary aggregator for X is dictatorial, then for every $n \geq 2$, every n-ary aggregator for X is locally monomorphic.

Proof Lemma 3.4

Proof.

The conclusion is true for binary aggregator
For induction, suppose the conclusion is true for all (n-1)-ary aggregator, where $n \geq 3$.
Consider an n-ary aggregator $\bar{f} = (f_1, ..., f_m)$ and pair of two-element subsets (B_i, B_j) where $B_i \subseteq X_i$ and $B_j \subseteq X_i$. T

Proof Lemma 3.4 Continue

Proof.

Let there be a column-vector $(a^1, ..., a^n)$ with $a^i \in \{0, 1\}$ with copies in B_i and B_j where $f_i(a^1, ..., a^n) \neq f_j(a^1, ..., a^n)$
As $n \geq 3$, by the pigeonhole principle, there is at least two position with the same element. Let these be the last two $a^n = a^{n-1}$. We then define a (n-1)-ary aggregator $\bar{g} = (g_1, ..., g_m)$ as follows: given $n - 1$ voting patterns $(x_1^i, ..., x_m^i), i = 1, ..., n - 1$, define n voting patterns by repeating the last one L

Proof.

Then for all $k = 1, ..., m$ define

$$g_k(x_k^1, ..., x^{n-1}) = f_k(x_k^1, ..., x^{n-1}, x^{n-1})$$

This shows that the (n-1)-ary aggregator is not locally monomorphic, which create a contradiction. □

Lemma 3.5

For $n \geq 2$, every n-ary aggregator $\bar{f} = (f_1, ..., f_m)$, there is an integer $d \leq n$ such that every integer $j \leq m$ and every two-element subset $B_j \subseteq X$, the restriction $f_i|B_j$ is equal to pr_d^n, the n-ary projection on the d-th coordinate.

For $n \geq 2$, and every n-ary aggregator $\bar{f} = (f_1, ..., f_m)$ and for all $s \geq 2$, there is an integer $d \leq n$ such that every integer $j \leq m$ and every subset $B_j \subseteq X_j$ of cardinality of at most s, the restriction $f_i|B_j$ is equal to pr_d^n

Proof

The induction basis for $s = 2$ is given.
For the inductive step, let $s \geq 3$ and assume the lemma is true for $s - 1$ where $f_j|B_j = pr^n_d$. Without loss of generality, we fix $d = 1$ for the projection. We also assume $s \leq n$.

Proof Continue

Towards a contradiction, let there exists $j_0 \leq m$ and row vector $a^1, ..., a^n$ in X where $B_{j_0} = \{a^1_{j_0}, ..., a^n_{j_0}\}$ has cardinality s and

$$f_{j_0}(a^1_{j_0}, ..., a^n_{j_0}) \neq a^1_{j_0}$$

By supportiveness, there exists $i_0 \in \{2, ..., n\}$ where

$$f_{j_0}(a^1_{j_0}, ..., a^n_{j_0}) \neq a^{i_0}_{j_0}$$

Without loss of generality, we fix $i_0 = 2$

Let $\{k_1, ..., k_s\}$ be a subset of $\{1, ..., n\}$ of cardinality s such that $\{a_{j_0}^{k_1}, ..., a_{j_0}^{k_2}\}$ are pairwise distinct.
If $i \notin \{k_1, ..., k_s\}$, then there is $l \in \{1, ..., s\}$ such that $a_{j_0}^{i} = a_{j_0}^{k_l}$
We then renumber $\{k_1, ..., k_s\}$ to $\{1, ..., s\}$.

Proof Continue

Let $B_{j_0}^{-} = \{a_{j_0}^1, ..., a_{j_0}^{s-1}\}$. We define an $(s-1)$-ary aggregator $\bar{f}^{-} = (f_1^{-}, ..., f_m^{-})$ for $j = 1, ...m$ and $(x_j^1, ..., x_j^{s-1}) \in X_j^{s-1}$.
We then define $(y_n^1, ..., y_j^n) \in X_j^n$ as:

$$y_j^i = \begin{cases} x_j^i & \text{for } i = 1, ..., s-1 \\ a_j^s & \text{if } i = s \\ a_j^s & \text{if } i > s \text{ and } a_{j_0}^i = a_{j_0}^s \\ x_j^l & \text{for the least } l < s \text{ such that } a_{j_0}^i = a_{j_0}^l, \text{ if } i > s \text{ and } a_{j_0}^i \neq a_{j_0}^s \end{cases}$$