Presentation for 21 February 2021

Social Choice Theory

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Conditions:

- Unrestricted Domain
- Non-Dictatorship
- Pareto Efficiency
- Independence of Irrelevant Alternatives

With Voters X, Y and Z

Case on a breach on contract

Judge	Contract?	Breach ?	Liable ?
Joe:	Yes	Yes	Yes
Judy:	Yes	No	No
Jules:	No	Yes	No

- 1) Binding contract ?
- 2) Breached ?
- 3) Liable ?

Case on a breach on contract

Judge	Contract?	Breach ?	Liable ?
Joe:	Yes	Yes	Yes
Judy:	Yes	No	No
Jules:	No	Yes	No

Judge	p	q	$p \wedge q$
Joe:	Yes	Yes	Yes
Judy:	Yes	No	No
Jules:	No	Yes	No
Majority:	Yes	Yes	No

Discursive Dilemma

Framework

- $\mathcal{L}$ - set of proposition
- Formula φ
- Complement $\varphi = \neg\varphi$
- Agenda $\Phi \subseteq \mathcal{L}$
- Example $\Phi = \{p, \neg p, q, \neg q, p \wedge q, \neg(p \wedge q)\}$

- Judgement set $J \subseteq \Phi$
- Example: $J_3 = \{\neg p, q, \neg(p \wedge q)\}$

Judge	p	q	$p \wedge q$
Joe:	Yes	Yes	Yes
Judy:	Yes	No	No
Jules:	No	Yes	No
Majority:	Yes	Yes	No

A judgement set J can be

- Complete: $\varphi \in J$ or $\sim \varphi \in J$ for all $\varphi \in \Phi$
- Complement-Free: Either φ or φ but not both
- Consistent
- Subset of complete and consistent judgement: $\mathcal{J}(\Phi)$

- Agents : $N = \{1, \ldots, n\}$
- Coalitions: $C \subseteq N$,
- Coalitions Complement: $\bar{C} := N \backslash C$
- Profile: $\mathbf{J} = (J_1, \ldots, J_n) \in \mathcal{J}(\Phi)^n$
- $N^{J}_{\varphi} := \{i \in N | \varphi \in J_i\}$

- Judgement Aggregation Rule: $f : \mathcal{J}(\Phi)^n \rightarrow 2^{\Phi}$
- Example: Majority Rule: $f_{maj} : \mathbf{J} \mapsto \{\varphi \in \Phi || N_{\varphi}^{J}| > \frac{n}{2}\}$

Technical Definition

- **Hamming Distance:** $H(J, J') := |J \backslash J'|$
- **Minimally Inconsistent set** X: every proper subset of X is consistent. (Note: X is a set of formulas)

Paradox where $a \succ b \succ c \succ a$

Joe:	$a \succ b \succ c$
Judy:	$c \succ a \succ b$
Jules:	$b \succ c \succ a$

	$P_{a \succ b}$	$P_{b \succ c}$	$P_{a \succ c}$
Joe:	Yes	Yes	Yes
Judy:	Yes	No	No
Jules:	No	Yes	No
Majority:	Yes	Yes	No

Properties of Aggregation Rules

A function can be

- unanimous
- anonymous
- neutral
- independent
- monotonic

Lemma 17.1 Winning Coalitions

Let f be an independent aggregator.
For $\varphi \in \Phi$, let $W_\varphi \subseteq 2^N$ be the winning coalition.
$\varphi \in f(J) \Leftrightarrow N^J_\varphi \in W_\varphi$ for all $J \in \mathcal{J}(\Phi)^n$
Then the following is true:

- f is unanimous iff $\forall \varphi \in \Phi(N \in W_\varphi)$
- f is anonymous iff
  $\forall C, C' \subseteq N \;\&\; \forall \varphi \in \Phi((C \in W_\Phi \;\&\; |C| = |C'|) \rightarrow C' \in W_\varphi)$
- f is neutral iff $\forall \varphi, \psi \in \Phi(W_\varphi = W_\psi)$

Winning Coalitions

- f is monotonic iff W_φ is upward closed
- f is complement-free iff $C \notin W_\varphi$ or $\bar{C} \notin W_\varphi$
- f is complete iff $C \in W_\varphi$ or $\bar{C} \in W_\varphi$

Theorem

No aggregator for an agenda of the form $\Phi\{p, q, p \wedge q\}$ can be anonymous, neutral, independent, complete and consistent.

Proof for List and Pettit

Proof.

Let the agenda be $\Phi \subseteq \{p, q, p \wedge q\}$. For contradiction let there be a function f that is anonymous, neutral, independent, complete, and consistent.

Consider a profile J in which there are 5 agents. 2 agents accepts p and q, 1 agent accepts p but not q, 1 agent accepts q but not p and the remaining agent accepts neither.

Proof.

We can observe that $|N^J_p| = |N^J_q| = |N^J{}_{\neg(p \wedge q)}| = 3$

As a result of being accepted by the same number of judges we must treat all three statement equally, either accepting all of them of rejecting all.

The former will result in the loss of consistency. The latter however would mean accepting $\neg p, \neg q$ and $(p \wedge q)$ which would also result in the loss of consistency.

Proof.

For generality we can simplify the number of agents that accepts p and q to $\frac{n-1}{2}$, the number of agents that accepts p but not q and vice versa to 1 and the remaining $\frac{n-3}{2}$ to accept neither p or q. □

Aggregation Rules: Quota-Based

A quota rule is the function f_q is induce by $q : \Phi \to \{0, 1, ..., n+1\}$, map formulas to thresholds:

$$f_q(J) = \{\varphi \in \Phi | \, |N^J_\varphi \geq q(\varphi)\}$$

What happens when quote is 0 or $n + 1$?

When all quotas are the same f_λ: Uniform Quota Rule

String Majority Rule: $f_{\frac{n+1}{2}}$

Intersection Rule: f_n

As we increase the quota, the less likely we are to obtain inconsistency.

Proposition 17.3 (Dietrich and List)

Let k be the size of the largest minimally inconsistent subset of the agenda Φ. Then every uniform quota rule f_λ with a quota of $\lambda > \frac{k-1}{k} \cdot n$ is consistent.

Proof

Proof.

For a contradiction, let there be a profile $J \in \mathcal{J}(\Phi)^n$ for which $f_\lambda(J)$ is inconsistent.

Let $X \subseteq f_\lambda(J)$ be a arbitrary minimally inconsistent subset. By assumption, $|X| \leq k$

Each formula φ had at least λ agents accepting it making the total acceptance to be $\lambda \cdot |X|$.

Proof.

As the $\lambda \cdot |X|$ acceptance have to come from n agents, by the pigeon hole principle, at least one of the agent have accepted at least $\frac{\lambda \cdot |X|}{n}$ of the formula.

But because $\lambda > \frac{k-1}{k}$, we can get $\frac{\lambda \cdot |X|}{n} > |X| - \frac{|X|}{k}$ and as $\frac{|X|}{k} \leq 1$, that one agent would have accepted at least $|X|$ formulas resulting in a consistency which is a contradiction. □