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Social Choice Theory

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Borda Rule aka Borda Count

Selects alternative with highest score

102	101	100	1	Symmetric	Asymmetric
a	b	c	c	1	2
b	c	a	b	0	1
c	a	b	a	-1	0

Symmetric Score:

a=(102*1)+(101*-1)+(100+0)+(1*-1)= 0

b=(102*0)+(101*1)+(100*-1)+(1*0) = 1

c=(102*-1)+(101*0)+(100*1)+(1*1) = -1 [winner is b; final scores differ by 1]

Asymmetric Score:

a = (102*2)+(101*0)+(100*1)+(1*0) = 304

b = (102*1)+(101*2)+(100*0)+(1*1) =305

c = (102*0)+(101*1)+(100*2)+(1*2) = 303 [winner is b; final scores differ by 1]

Borda Rule

Hypothesis: The symmetric borda score and asymmetric Borda score induce the same SCF (Winner)

Proof.

Given a vector of scoring weights for a for an odd number of m alternative for the symmetric Borda Score $\{w_1, w_2, ..., w_m\}$ where $w_1 = -w_m, w_2 = -w_{m-1}$...,with the relationship between weights being $w_i > w_{i+1}$ and $w_{\frac{m+1}{2}} = 0$ with a difference between each consecutive weight of $w_i - w_{i+1} = d$. Let the value of $w_1 = b$. A vector of scoring weights $\{b, b, ..., b\}$ will give all alternatives nb score for n voters.

□

Proof.

An asymmetric vector scoring weights of similar difference d can be constructed by adding a vector of length m of value b to the weights obtaining $\{w_1 + b, w_2 + b, ..., w_m + b\}$ which gives us the relationship between the asymmetric score and symmetric score for n voters of $Borda_p^{asym}(x) = bn + Borda_p^{sym}(x)$ and as bn is the same for all alternative, the asymmetric Borda score will give the same winner as the symmetric Borda score with the same alternative with the highest score. □

Copeland's Rule

$Copeland(x) = |\{y \in A | x >^{\mu} y\}| - |\{y \in A | y >^{\mu} x\}|$

Compare every two pairwise alternative and order them by the number of total victories with the winner being the alternative with the most pairwise victories.

10
a
b
c

Copeland(a) = 2

Copeland(b) = 0

Copeland(c) = -2

Winner = a

Definition

A Condorcet winner for a profile P is an alternative x that defeats **every other** alternative in the strict pairwise majority sense: $x >_p^\mu y$ for all $x \neq y$. Condorcet winner can be found using the Pairwise Majority Rule, PMR.

Definition

A Condorcet cycle can exist when there is three or more alternatives where the collective preference is cyclic. For example, A is preferred over B, B is preferred over C and yet C is preferred over A. No PMR winner exist.

1	1	1
a	b	c
b	c	a
c	a	b

$a >^\mu b, b >^\mu c, c >^\mu a$

Definition

An SCF f is a Condorcet extension or is Condorcet consistent if f selects the Condorcet winner alone for each profile $P \in D_{Condorcet}$ where $D_{Condorcet}$ is the set of all profiles for which a Condorcet winner exists.

Condorcet Extension: Borda and Copeland

3	2
a	b
b	c
c	a

The Condorcet winner for the profile is a as it defeats b and c in a pairwise comparison. But using an asymmetric Borda scoring vector of $w = (2, 1, 0)$ resulting in Borda(a)=6,Borda(b)=7,Borda(c)=1, which results in b being the Borda winner.

Borda is not a Condorcet extension because it uses the net difference across alternatives.

Copeland is a Condorcet extension because it uses pairwise comparison.

Cambell-Kelly Theorem

Consider SCFs with domain $D_{Condorcet}$ for three or more alternatives. Pairwise Majority Rule is resolute, anonymous, neutral, and strategyproof; for an odd number of voters, *it is the unique such rule*.

Proof.

Suppose that PMR is not resolute with two ore more winners. This would be impossible as such profiles would not be in the domain $D_{Condorcet}$. PMR is strategyproof. Suppose voter i's (sincere) ballot has $y \succ_i x$ with x being the Condorcet winner, there is no way to cast an insincere vote to reverse $x >^{\mu} y$. [not quite clear on anonymous and neutral] □

Simpson Rule

$Simpson_p(x) = min\{Net_p(x > y) | y \in A \backslash \{x\}\}$

The Simpson rule selects as the winner the candidate whose greatest pairwise defeat is smaller than the greatest pairwise defeat of any other candidate.

3	2
a	b
b	c
c	a

Simpson(a)=1

Simpson(b)=-1

Simpson(c)=-1

It is a Condorcet extension because it compares pairwise and only take the minimum pairwise.

Fix some enumeration $\{x1, x2, ..., xm\}$ of the alternatives. The winner of round 1 is $x1$; the winner of round $i+1$ is the winner w of round i, if $w \geqslant^{\mu} x_{i+1}$ and is x_{i+1} if $x_{i+1} \geqslant^{\mu} w$; the ultimate winner is the winner of round m.

This rule is a Condorcet extension.

Axioms 2: Reinforcement and Monotonicity Properties

How to respond to:

- One or more voters modifying their ballots
- One or more voters are added to a profile

Definition

Reinforcement requires that the common winning alternatives chosen by two disjoint set of voters be exactly those chosen by the union of these sets;precisely, an SCF f is reinforcing if $f(s) \cap f(t) \neq \emptyset \rightarrow f(s+t) = f(s) \cap f(t)$ for all voting situations s and t

Weak form: Homogeneity
$f(ks) = f(s)$ for each $k \in \mathcal{N}$
Intermediate form:
$f(s) = f(t) \rightarrow f(s+t) = f(s) = f(t)$

Reinforcement

Scoring rules are reinforcing if for some alternative x has highest score for s and t both, the x's score for s+t must also be highest
Compound Scoring: Any ties resulting from a first score vector may be broken by subsequent score vector.

Theorem

The anonymous,neutral, and reinforcing SCFs are exactly the compound scoring rules.